Definability aspects of ultrafilters

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 \mathcal{U} is closed under finite modification $\Rightarrow \mathcal{U}$ has measure 0 or 1, meager or comeager. But \mathcal{U} and $\{\omega \setminus U : U \in \mathcal{U}\}$ partition $\mathcal{P}(\omega) \Rightarrow$ contradiction.

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- **③** Analytic filters are bounded, i.e. contained in a σ -compact $\subseteq [\omega]^{\omega}$.
- There is a Σ_2^1 ultrafilter in *L*.
- **③** In fact any Σ_n^1 ultrafilter is already Δ_n^1 ($U \in \mathcal{U}$ iff $\omega \setminus U \notin \mathcal{U}$).

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Recall that X is a base for \mathcal{U} iff $\mathcal{U} = \{U : \exists V \in X(V \subseteq U)\}.$

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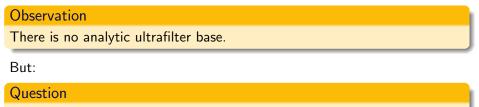
Note that an analytic base generates an analytic filter. Thus

Observation

There is no analytic ultrafilter base.

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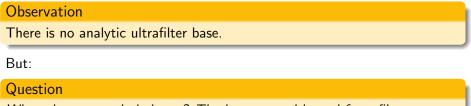
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A. Miller streamlined a technique for constructing various combinatorial families of reals in a Π_1^1 way in L. For instance he constructed a Π_1^1 mad family, independent family, Hamel basis, ... in L.

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Using this technique we could show the following:

Theorem	
(V=L)	
1	There is a Π^1_1 P-point base.
2	There is a Π^1_1 Q-point base.

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In strong contrast we showed:

Theorem

There is no Π_1^1 base for a Ramsey ultrafilter.

Recall that \mathcal{U} is Ramsey iff \mathcal{U} is a P- and a Q-point.

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Main ingredients:

Lemma

Let \mathcal{U} be Ramsey and $M \preccurlyeq H(\theta)$ countable where $\mathcal{U} \in M$. Then there is $x \in \mathcal{U}$ so that every $y \subseteq x$ is generic over M.

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Lemma

Whenever X is Π_1^1 , $Y \subseteq X$ with $\sup\{\omega_1^y : y \in Y\} < \omega_1$, then there is $Y' \Delta_1^1$ such that $Y \subseteq Y' \subseteq X$.

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Theorem (Shelah)

(GCH) Let \mathcal{U} be an arbitrary Ramsey ultrafilter. Then there is a forcing extension in which \mathcal{U} generates the unique (up to permutation of ω) *P*-point. Moreover it still generates a Ramsey ultrafilter.

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Corollary

It is consistent that every P-point is Ramsey and Δ_2^1 (in particular there is no Π_1^1 base for a P-point).

Still we have that

Theorem

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In fact, whenever \mathcal{U} is Δ_2^1 , then $\mathcal{U} \otimes \mathcal{U}$ has a Π_1^1 base.

 $\mathcal{U}\otimes\mathcal{U}=\{x\subseteq\omega\times\omega:\{n\in\omega:\{m\in\omega:(n,m)\in x\}\in\mathcal{U}\}\in\mathcal{U}\}.$

$$\mathfrak{x} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \text{ is a } \dots \text{ family}\}$$

then

$$\mathfrak{x}_{\mathcal{B}} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \Delta^1_1, \bigcup \mathcal{B} \text{ is a } ... \text{ family}\}$$

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Observation

$$\begin{split} \aleph_1 &\leq \mathfrak{u}_B \leq \mathfrak{u} \leq \mathfrak{c}. \text{ In fact } \mathfrak{b} \leq \mathfrak{u}_B. \\ \exists \Delta_2^1 \text{ ultrafilter } \Rightarrow \mathfrak{u}_B = \aleph_1. \end{split}$$

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Question

Is $u_B < u$ consistent?

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Question

Is $\mathfrak{u}_B < \mathfrak{u}$ consistent?

How to make \mathfrak{u} large?

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Is $u_B < u$ consistent?

How to make \mathfrak{u} large? Add splitting reals!

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Is $\mathfrak{u}_B < \mathfrak{u}$ consistent?

How to make u large? Add splitting reals! But without adding dominating reals!

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Is $\mathfrak{u}_B < \mathfrak{u}$ consistent?

How to make $\mathfrak u$ large? Add splitting reals! But without adding dominating reals!

Classical forcing notions doing this are: Cohen, Random and Silver forcing.

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Theorem

Cohen/Random/Silver forcing adds a real that is splitting over any OD(V)-definable filter (e.g. projective coded in V).

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Corollary

After adding ω_1 many Cohen/Random/Silver reals there is no projective ultrafilter.

Question

Is it possible to preserve an ω_1 -Borel ultrafilter while adding a splitting real?

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- \mathbb{P} has the mutual genericity property.

Definition

 \mathbb{Q} has the mutual genericity property iff for any $M \preccurlyeq H(\theta)$ countable, $p, \mathbb{Q} \in M$, there is a master condition $q \le p$ so that for any filters G_0, \ldots, G_k containng q, generic over M and pairwise different over M,

 $G_0 \times \cdots \times G_k$ is \mathbb{Q}^{k+1} generic over M.

Example: Sacks forcing, easy fusion argument.

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Theorem

(CH) There is a collection \mathcal{A} of Borel sets so that for any Suslin forcing \mathbb{Q} with the mgp, $V^{\mathbb{Q}} \models \bigcup \mathcal{A}$ is an ultrafilter. (V=L) There is a (lightface) Π_1^1 ultrafilter base X so for any ... $V^{\mathbb{Q}} \models X$ is an ultrafilter base.

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It is possible to add a splitting real while preserving a Π_1^1 ultrafilter base.

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Corollary

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Remark

The theorem also applies to maximal independent and maximal almost disjoint families.

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